Notes on Problem Set 8

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Below is a problem set I created for a microeconomics course from Fall 2015, for which I was the teaching assistant.

I want to explain the reasoning behind this problem set and put it in the context of the typical approach in economics. Here is a sample from an actual problem set for the same course that is typical of the field:

Problem 1: In the market for a certain commodity, demand is given by $Q = 250 - 5P$ and supply is given by $Q = 5P$.

(a) What is the equilibrium price and quantity?

(b) What are consumer and producer surplus?

This problem set simply asked the student to run through the math of solving a basic system of two equations then copy what was done in the book for calculating a number and calling it “consumer surplus”. It does not prompt any understanding of what consumer surplus is, how the market might be working, or what assumptions underlie the analysis.

In contrast, my problem set proceeds as follows:

- I start out by repeating a portion of a previous problem (Katherine had been seen before and they had previously done parts a&b). This allows the students to see the connection to previous material
- Next (through I.1) I have the students calculations dollar amount we would have to give Katherine to make her indifferent between having the extra cash and being able to buy her daily coffee at its current price
- Students then expand that to multiple consumers (I.2)
- And demonstrate to themselves how two different ways of calculating consumer surplus produce the same result
- In Part II the students solve a related example which more directly demonstrates how that concept is used in the real world
- Finally, in part IV they observe some limitations of that approach that stem from the underlying assumptions

Through this problem set students have learned techniques but also have gained a deeper understanding of the material and connection of it with the world.

Problem Set 8 - Solutions

Consumer Surplus & Taxes

This problems are harder and more involved than what you can expect to see on the test. That being said, they are good practice and hopefully help solidify some of the necessary concepts.

Also, in this problem set I ignore the nuanced difference between consumer surplus, compensating variation and equivalent variation. If you are interested a good discussion is here: `http://wps.aw.com/bp_perloff_microecon_6/180/46080/11796533.cw/content/index.html`, but for the exams in this class you will not need to know the fine differences.

Problem I: A consumer's surplus

1. *Katherine has utility over coffee and all-other-goods (Marshallian money) :* $u_{Katherine}(all-other-goods, coffee_{oz}) = u(y, q_c) = y + 4.8 * q_c^{0.25}$. *The price of all other goods is \$1 (it's a numeraire good, or Marshallian money denoted by y) and the price of a cup of coffee is* p_c. *She has* $\$M_k$ *dollars to spend.*

   (a) *Solve her utility maximization problem. Your answer should have her choice variables (consumption of coffee and consumption of all-other-goods) as functions of the exogenous variables (her money (*$\$M_k$*) and the price of coffee* p_c*).*

   This is the same set up we had in problem set 4:

   Write her constrained utility maximization problem, and then write down the Lagrangian for the utility maximization problem.

$$\max_{x_c, x_p} u(y, x_c) = y + 4.8 * x_c^{1/4}$$

$$s.t. M = y + x_c p_c$$

$$\mathcal{L} = y + 4.8 * x_c^{1/4} - \lambda(y + x_c p_c - M)$$

   Write down the three first order conditions...

$$\partial\mathcal{L}/\partial x_c = \frac{4.8}{4} x_c^{-3/4} - \lambda p_c = 0$$

$$\partial\mathcal{L}/\partial y = 1 - \lambda = 0$$

$$\partial\mathcal{L}/\partial\lambda = -(y + x_c p_c - M) = 0$$

   and solve the utility maximization problem.

   From $\partial\mathcal{L}/\partial y$ we get that $\lambda = 1$. Using that in our equation from $\partial\mathcal{L}/\partial x_c$ yields $1.2x_c^{-3/4} = p_c$, which means $x_c^* = (p_c/1.2)^{-4/3} = (1.2/p_c)^{4/3}$

   Katherine spends $x_c^* = p_c\big((1.2/p_c)^{4/3}\big)$ on coffee and use the rest of her money $(M - p_c\big((1.2/p_c)^{4/3}\big))$ for all-other-goods.

(b) *With the price of coffee at \$0.15 per ounce. How much coffee does she consume? How much AOG? What is Katherine's utility? (Note: these all should be a function of M_k only). Assume M_k is sufficiently large that at optimum Katherine buys some of both goods.*

At this price the amount of coffee Katherine would like to buy is given by $x_c^*(p_c, M) = (p_c/1.2)^{-4/3}$ so $x_c^*(0.15, M) = (0.15/1.2)^{-4/3} = 16$ ounces. 16 ounces cost 16*\$0.15 = \$2.40, and she spends the rest of her money ($M - \$2.40 > 0$) on all-other-goods.

Her utility is $u(y, q_c) = y + 4.8 * q_c^{0.25} = (M - \$2.40) + 4.8 * (16)^{1/4} = (M - \$2.40) + 4.8 * (16)^{1/4} = (M - \$2.40) + 9.6 = M + 7.2$

(c) *Now assume that either the price of coffee is infinity (and won't be bought), or equivalently coffee is totally unavailable (and can't be bought). How much coffee does she consume? How much AOG? (Note: It might be easier to not use your equations from (a), and instead just use logic/intuition) What is Katherine's utility? Assume M_k is sufficiently large that at optimum Katherine buys some of both goods.*

She just spends all her money, M, on all other goods. This gives her the utility $u(M, 0) = M + 4.8 * 0^{0.25} = M$.

(d) *With the price still infinite (or coffee not being available), what increase in his income would give him the same utility as in (b)? Does this depend on M_k (assuming, again that M_k is sufficiently large that at optimum Katherine buys some of both goods)?*

The difference between utilities is $(M + 7.2) - M = 7.2$. So she would need 7.2 extra money (which she'd buy other goods).

(e) *What you just solved for is Katherine's "net" consumer's surplus - the benefit (converted to monetary units) Katherine is getting by having the good available at the price of p_c compared to it not being available. Her utility from coffee (the benefit), minus what she paid for the coffee (the cost).*

Katherine's "gross" consumer surplus is just the benefit of having the good available at the price of p_c without considering what she paid (the cost). What is Katherine's gross consumer surplus?

She paid \$2.40 for the coffee, so her gross surplus is 7.2+2.4=9.6.

Another method of viewing gross surplus is to just look at the utility from her 16 ounces of coffee: $4.8 * (16)^{1/4} = 9.6$.

As a technical note, this method gives us the compensating variation. However, with quasi-linear preferences there are no income effects for price changes and CV=EV=CS.

2. Multiple consumers

   Now, assume there are N consumers with preferences identical to Katherine. They differ in the amount of money they have. Consumer i has money M_i. Assume that for each consumer M_i is sufficiently large that at optimum consumer i buys some of both goods.

   (a) *With price still at 0.15, what is the ("net") surplus for consumer i? What is the combined ("net") consumer**s**' surplus for all N consumers?*

   The net surplus for any consumer with identical preferences (as long as $M > 2.4$) will be 7.2. The combined surplus will be N*7.2.

   (b) *Why was the qualification that "M_i is sufficiently large that at optimum consumer i buys some of both goods" necessary?*

   If a consumer is unable to buy the full amount of coffee desired at that price (16 ounces) then they will have a smaller surplus from the availability of coffee.

   (c) *What is consumer i's demand function for coffee (Using p_c because we want this to be for any price)? (Hint: you solved for it in (1a)). Express this as the inverse demand function (price as a function of quantity). What is the combined (i.e. market) demand function for all N consumers? Hint: holding price constant, what would demand be if there were two Katherines?*

   Her demand function is $x_c^*(p_c, M) = (p_c/1.2)^{-4/3}$ and her inverse demand function is $x_c^{-3/4} = p_c/1.2$ or:

$$p_c = 1.2 * x_c^{-3/4}$$

   Let X be the market quantity. If there were two Katherines at the same price quantity would double: $X = 2 * x_c = (p_c/1.2)^{-4/3}$, or $X/2 = (p_c/1.2)^{-4/3}$ and $p_c = 1.2 * (X/2)^{-3/4}$. Similarly, with N consumers the quantity is multiplied by N: $X = N * x_c = (p_c/1.2)^{-4/3}$. So $(X/N)^{-3/4} = p_c/1.2$ or:

$$p_c = 1.2 * (X/N)^{-3/4}$$

   (d) *What is the integral of the inverse market demand function over all quantities (from 0 to Q=16*N)? Hint:*

$$\int_0^Q \left(\frac{x}{16}\right)^{-3/4} dx = 4 * (16^{3/4}) * (x)^{1/4}\Big|_0^Q = 4 * 8 * (Q)^{1/4} - 4 * 8 * (0)^{1/4} = 32 * \left(Q^{1/4}\right)$$

   Using our function from (e):

$$\int_0^{Q=N*16} 1.2 * \left(\frac{X}{N}\right)^{-3/4} dx = 1.2 * 4 * N^{3/4} * (X)^{1/4}\Big|_0^{N*16}$$

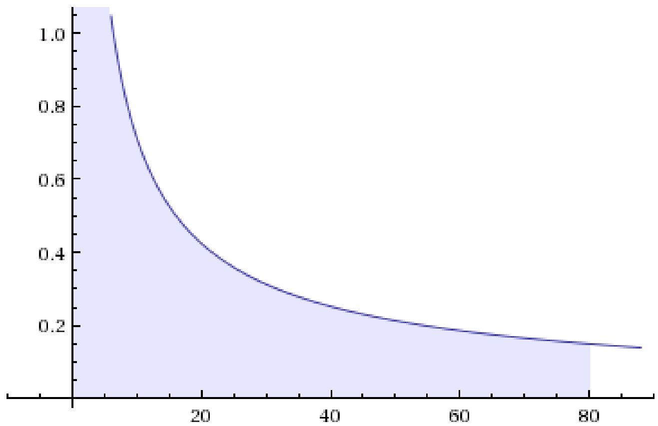
$$= 4.8 * N^{3/4} * (N * 16)^{1/4} - 4.8 * N^{3/4} * (0)^{1/4}$$

$$= 4.8 * N^{3/4} * (N)^{1/4} * (16)^{1/4} = 4.8 * N * 2 = N * 9.6$$

Definite integral:

$$\int_0^{80} 1.2 \left(\frac{x}{5}\right)^{-3/4} dx = 48.$$

Visual representation of the integral:



Shown for N=5.

So with our assumptions the ("gross") consumer surplus just scales linearly, and in this case both methods for calculating consumer surplus ("area under the demand curve" and "extra utility from having it available at a certain price") produce the same answers.

This is true because of the quasi-linear preferences, because there is no income effect CV=CS. But generally the two measures will be very close, and the concept they are trying to capture is quite similar.

(e) *Combining all of the consumers, how much is spent on coffee?*

Each spends 2.4, so the total spent is 2.4*N.

If you successfully navigated this problem, then you should see that (d)-(e)=(a)!

$N * 9.6 - N * 2.4 = N * 7.2$. So, yes! Both methods, in this case, produce the same total net consumers' surplus.

Problem II: Cost-Benefit Analysis

1. The government wants to build a bridge. It assumes all consumers have (daily*) quasi linear demand for bridge use in the same form: $u(AoG, Bridge) = u(y, q_b) = y + q_b^{0.5}$.

   (*By daily I mean that the utility depends on how times they cross in a given day. The utility is then summed across days, and for money purposes you can assume that every day is the same, and some how much they pay)

   (a) *First, solve the utility maximization problem for one consumer. This will show how much the bridge will be used at a given price (toll).*

   Very similar to I.1.a:

   Write the constrained utility maximization problem, and then write down the Lagrangian for the utility maximization problem.

$$\max_{y,q_p} u(y, q_b) = y + q_b^{1/2}$$

$$s.t. M = y + p_b q_b$$

$$\mathcal{L} = y + q_b^{1/2} - \lambda(y + p_b q_b - M)$$

   Write down the three first order conditions...

$$\partial\mathcal{L}/\partial q_b = \frac{1}{2} q_b^{-1/2} - \lambda p_b = 0$$

$$\partial\mathcal{L}/\partial y = 1 - \lambda = 0$$

$$\partial\mathcal{L}/\partial\lambda = -(y + p_b q_b - M) = 0$$

   ...and solve the utility maximization problem:

   From $\partial\mathcal{L}/\partial y$ we get that $\lambda = 1$. Using that in our equation from $\partial\mathcal{L}/\partial q_c$ yields $\frac{1}{2} q_b^{-1/2} = p_b$, which means $q_b^* = 1/\left(4 * p_b^2\right)$

   The consumer will spend $p_b * x_b^* = p_b * 1/\left(4 * p_b^2\right) = 1/\left(4p_b\right)$ on crossing the bridge and use the rest of her money, $M - 1/\left(4p_b\right)$, for all-other-goods.

   (b) *If someone is only going to use the bridge once per **day** (year) ($q_b = 1$), what is the most they would pay (each **day** (year))? Hint: use the inverse demand function.*

   The problem set said "year", I meant day. For once per day: from (a) we have $q_b^* = \frac{1}{4} p_b^{-2}$, so $\left(4q_b\right)^{-1/2} = p_b$. If we want $q_b = 1$ then $p_b = 0.5$

(c) *If the bridge is estimated to last 20 years and cost $730 million to build, how many consumers (all identical) would there have to be in order for it to be worth building?*

If each consumer uses the bridge once per day, they use it 365 days a year and 7,300 times over 20 years. If they are willing to pay $0.50 for each day (time that they cross), the consumer is willing to pay 7,300*$0.50 = $3,650 over the life of the bridge.

We then would need $730 million / $3,650 = 200,000 people with identical preferences (crossing the bridge once a day).

If you did once per year it would just be 20,000*365 = 7,300,000.

Problem III: Different Utility

1. *Sam has utility over burgers and all-other-goods (y). However, unlike quasi-linear utility Sam's utility for all-other-goods faces diminishing marginal returns. He has the following utility function:* $u_{Sam}(AoG, burgers) = u(y, q_b) = 4 * y^{0.25} + 4 * q_b^{0.25}$. *The price of all other goods is \$1 (it's a numeraire good, or Marshallian money denoted by y) and the price of a burger is denoted* p_b. *He has* $\$M_k$ *dollars to spend.*

   (a) *Solve his utility maximization problem. Your answer should have his choice variables (consumption of burgers and consumption of all-other-goods) as functions of the exogenous variables (his money (*$\$M_k$*) and the price of burgers* p_b*).*

   This is different from other utility functions that we have seen (and the algebra can get a bit messy) but the process is very similar:

   Write the constrained utility maximization problem, and then write down the Lagrangian for the utility maximization problem.

$$\max_{y,q_p} u(y, q_b) = 4 * y^{1/4} + 4 * q_b^{1/4}$$

$$s.t. M = y + p_b q_b$$

$$\mathcal{L} = 4 * y^{1/4} + 4 * q_b^{1/4} - \lambda(y + p_b q_b - M)$$

   Write down the three first order conditions...

$$\partial\mathcal{L}/\partial q_b = q_b^{-3/4} - \lambda p_b = 0$$

$$\partial\mathcal{L}/\partial y^{3/4} = y^{-3/4} - \lambda = 0$$

$$\partial\mathcal{L}/\partial\lambda = -(y + p_b q_b - M) = 0$$

   ...and solve the utility maximization problem:

   From $\partial\mathcal{L}/\partial y$ we get that $\lambda = 1$.

   Using that in our equation from $\partial\mathcal{L}/\partial q_c$ (putting λp_b on the right hand side) and dividing it by our equation from $\partial\mathcal{L}/\partial y$ (putting λ on the right hand side) yields $\frac{q_b^{-3/4}}{y^{-3/4}} = p_b$, which means $\frac{q_b}{y} = p_b^{-4/3}$ and gives us the relationship between q_b and y: $q_b = y * p_b^{-4/3}$.

   Substituting this relationship for q_b in the budget constraint yields $M = p_b * \left(y * p_b^{-4/3}\right) + y = y * p_b^{-1/3} + y = y * (1 + p_b^{-1/3})$. This means that:

$$y^* = M/(1+p_b^{-1/3}) = M\left(\frac{1}{1+\frac{1}{p_b^{1/3}}}\right) = M\left(\frac{1}{\frac{p_b^{1/3}+1}{p_b^{1/3}}}\right)$$

$$= M\left(\frac{p_b^{1/3}}{p_b^{1/3}+1}\right)$$

Substituting this into the relationship between y and q_b give us:

$$q_b^* = M\left(\frac{p_b^{1/3}}{p_b^{1/3}+1}\right) * p_b^{-4/3} = M\left(\frac{p_b^{-1}}{p_b^{1/3}+1}\right) = \frac{M}{p_b}\left(\frac{1}{1+p_b^{1/3}}\right)$$

As an observation, the consumer will spend $p_b * x_b^* = p_b * \frac{M}{p_b}\left(\frac{1}{1+p_b^{1/3}}\right) = M\left(\frac{1}{1+p_b^{1/3}}\right)$ on burgers and $M\left(\frac{p_b^{1/3}}{1+p_b^{1/3}}\right)$ on all other goods.

(b) *With the price of a burger at \$8 how many burgers does he consume? How much AOG? What is his utility? (Note: these all should be a function of M_k only).*

$y^*(p_b) = M\left(\frac{p_b^{1/3}}{p_b^{1/3}+1}\right)$ so $y^*(8) = M\left(\frac{8^{1/3}}{8^{1/3}+1}\right) = M\left(\frac{2}{2+1}\right) = \frac{2}{3}M$

And $q_b^*(p_b) = \frac{M}{p_b}\left(\frac{1}{1+p_b^{1/3}}\right)$ so $q_b^*(8) = \frac{M}{8}\left(\frac{1}{1+8^{1/3}}\right) = \frac{M}{8}\left(\frac{1}{1+2}\right) = \frac{1}{8}\left(\frac{M}{3}\right)$.

His utility is $u(y, q_b) = 4 * y^{1/4} + 4 * q_b^{1/4}$ so with the optimal choices at \$8 a burger his utility is: $u(M*2/3, M/(3*8)) = 4*\left(\frac{2}{3}M\right)^{1/4} + 4*\left(\frac{1}{8}\left(\frac{M}{3}\right)\right)^{1/4} = (M^{1/4})4\left(\left(\frac{2}{3}\right)^{1/4} + \left(\frac{1}{24}\right)^{1/4}\right) \approx 5.4*(M^{1/4})$

(c) *Now assume that either the price of burgers is infinity (and won't be bought), or equivalently burgers are totally unavailable (and can't be bought). How many burgers does he consume? How much AOG? (Note: It might be easier to not use your equations from (a), and instead just use logic/intuition) What is Sam's utility?*

He buys not burgers and spends all M on other goods. So his utility is: $u(M, 0) = 4*(M)^{1/4} + 4*(0)^{1/4} = 4*(M)^{1/4}$

(d) *With the price still infinite (or burgers not being available), what increase in his income would give him the same utility as in (b)? Does this depend on M_k?*

Want to find the the extra money he needs $(M_2 - M)$, where M_2 is the money he would have that gives equal utility. That means that $5.4 * (M^{1/4}) = 4 * (M_2)^{1/4}$
$1.35 * (M^{1/4}) = (M_2)^{1/4}$
$3.375 * M = M_2$

So the extra money needed is $3.375 * M - M = 2.375M$ which depends on M.

Again, this is technically the "compensating variation" not consumer surplus. But consumer surplus will also depend on M (and is harder to calculate).

(e) *What does your answer in part (d) tell us about being able to easily aggregate surpluses across consumers?*

With this type of utility we cannot easily aggregate across consumers without knowing their income, or at least their income distribution.

(f) *What are some potential disadvantages (in a normative sense) to assuming quasi linear utility in performing cost-benefit analysis?*

When we ignore differences in incomes and differences in we may not be valuing the public good correctly.

Furthermore, a huge problem is that while the benefits accrue differently based on wealth (in this example someone with twice the income receives twice the benefit) usually there is no way to tax differently. In this example, rich and poor would pay the same but rich would receive more utility.